1 Hidden Charge

1.1 Finding x_Q and y_Q

The first step is to locate the x and y coordinates of the test charge. Two approaches are illustrated here.

1.1.1 Method 1

Select any initial launch point, and keep it fixed. $(x_i, y_i) = (0, 0)$ is a good choice. Vary the accelerating voltage in order to obtain several screen hits; plot these on a graph. Draw a line through the points, extended in both directions. The target charge must lie on this line.

Repeat with a different launch point. $(x_i, y_i) = (0, 10)$ is a good choice. The two lines will intersect, this is likely the location of the target charge.

Select a third launch point, one that would be located approximately perpendicular to either of the first two lines. $(x_i, y_i) = (0, -10)$ is a good choice. All three lines should intersect at a single point; that's the location of the target charge, (x_Q, y_Q)

1.1.2 Method 2

This method is much less accurate. Select a fixed value for x_i, and vary y_i. Observe y_f. There will be a value of y_i such that y_f is almost the same, while on either side of it, y_f will shift away from y_i. This special value such that $y_i \approx y_f$ is the location y_Q. Repeat the process with a fixed y_i and a varying x_i. Not that this technique won't work if the target charge is outside the bounds of the screen!

A student using this method cannot get full marks for the problem.

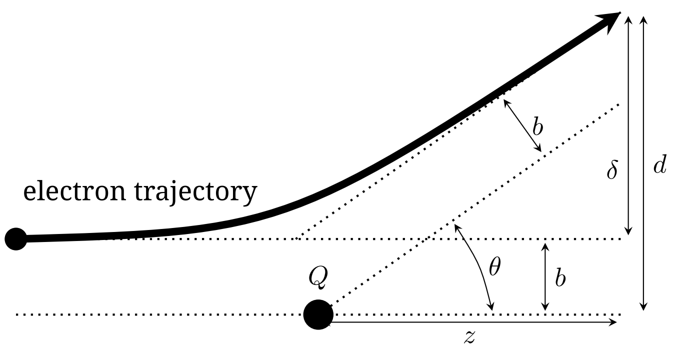
1.2 Determining Q and z_Q

Focus on the Rutherford equation. It is convenient to write it in the form

$$\tan\frac{\theta}{2} = \frac{kqQ}{2Eb}$$

One choice is to try and keep θ fixed, and vary E and b. The other choice is to keep b fixed and small, and vary E. Each approach has benefits and drawbacks.

The Rutherford scattering equation diagram can also be drawn as below



1.2.1 Method 1: keep θ fixed

The screen distance d is given by the relation

$$d\cos\theta = z\sin\theta + b$$

This is only strictly true if the electron has reached the scattered asymptote, otherwise, the measured value of d will be larger than the true value.

For a fixed value of θ, which will happen if the product bE is kept constant, graph b vertically against d horizontally. The slope of the graph will yield the value of $\cos\theta$, the intercept will yield $z\sin\theta$. The sign of z is unimportant, as it was implied that it is behind the screen. Return to Rutherford's equation to find Q.

One can improve the results by selecting values of b that are small, as this forces the electron to be closer to the scattered asymptote.

Method 1A: focus on δ

One might think that it is easier to focus on the quantity δ, as it is directly measurable. The problem is that the intercept between the two asymptotes of the trajectory is not a distance z from the screen, it is farther by an amount $b/\tan(\theta/2)$. Neglecting this correction will lead to $\delta = z\tan\theta$ from which the student can only find the product zQ.

Including the correction yields a really ugly looking expression

$$\frac{\delta}{\tan\theta} = z + \frac{b}{\tan(\theta/2)}$$

Most efforts to reduce this expression will return the student to some equivalent of the previous paragraph, with only a value of zQ obtainable.

1.2.2 Method 2: keep b fixed and small

A second approach is to use the twice angle formula for the tangent:

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Let $2\alpha = \theta$, and then combining with the Rutherford equation,

$$\tan\theta = \frac{2\gamma E}{\gamma^2 E^2 - 1}$$

where $\gamma = 2b/kqQ$. If b is small compared to d, then $\tan\theta \approx d/z$. Combining, one gets the linear equation

$$\frac{2E}{d} = \frac{\gamma}{z}E^2 - \frac{1}{z\gamma}$$

Plotting $2E/d$ vertically against E^2 horizontally ought yield a straight line with a slope γ/z and an intercept $1/z\gamma$. The challenge here is the Gaussian error in the initial beam location; as b gets smaller that relative error becomes more significant. If there were no initial beam spread, then this approach would be exact in the limit $b \to 0$.

Method 2A: focus on δ

As before, this approach has the disadvantage that the intercept point of the trajectory asymptotes is a function of b and θ. Neglecting this error, a student could graph $2E/\delta$ vertically against E^2 horizontally. The student has gained some in that the error of $b/\cos\theta$ has been removed from the expression for d, but an error of $b/\tan(\theta/2)$ has been added to the expression for z. In this case, small angles are bad.

Still, if there were no initial beam spread, this approach would be exact in the limit $b \to 0$.

1.2.3 Method 3: finding only the product zQ

It is tempting to start with the approximation $\tan(\theta/2) = \delta/2z$. Doing so reduces Rutherford's formula to

$$\delta = \frac{kqQz}{Eb}$$

A student could keep δ fixed (though that's hard!), E fixed, or b fixed, and plot the appropriate combinations of the remaining two variables to get a straight line. From this they can deduce the product Qz.

A student would also arrive at this point by neglecting the intercept of method 2. In fact, since that intercept is extrapolated and sensitive to error, it is likely to have the wrong sign, and in that case the student has effectively ended up here.

1.3 Grading Schemes

1.3.1 Part 1 (1 p)

Attempting to locate x_Q and y_Q

Finding x and y: Method 1	
At least 3 lines (0.5p), only 2 lines (0.3p), only 1 line (0.1p)	0.5 p
At least 4 data points on each line (0.3p); at least 3 points each line (0.2p); at least 2 points on each line (0.1p). The initial point can be one the data points.	0.3 p
Points spread to fill ≈1/3 the plot on each line (0.2p); fill ≈1/5 on each line (0.1p)	0.2 p
Total possible for part 1:	**1.0 p**

If a student elects to solve the intersection of two (or more) lines, when each line has only two data points, then they get (0.4p) for two lines, (0.6p) for three, (0.7p) for four, and (0.8p) for five or more. Then assess the spread condition.

Finding x and y: Method 2	
Using method 2	0.4 p
Total possible for part 1:	**0.4 p**

1.3.2 Part 2 (3 p)

Accuracy of result for X_Q and y_Q

Finding x and y: Both Methods	
x in range $5.3 \to 5.5$ cm (1.0p); in range $5.2 \to 5.6$ cm (0.7p); in range $5.1 \to 5.7$ cm (0.4p); in range $5.0 \to 5.8$ cm (0.1p)	1.0 p
y in range $-2.5 \to -2.7$ cm (1.0p); in range $-2.4 \to -2.8$ cm (0.7p); in range $-2.3 \to -2.9$ cm (0.4p); in range $-2.2 \to -3.0$ cm (0.1p)	1.0 p
x and y values each within one student stated error of $(5.4, -2.6)$ (0.5p); within one stated error for one value, and within two stated errors for the other (0.3p); within two stated errors for both values (0.2p); within two stated errors for one value (0.1p)	0.5 p
Statement of error for both x and y clearly reflected and consistent in graphical picture or math (0.5p); statement of error only concerned with 1mm screen resolution or 1 mm beam resolution (0.1 p) for each	0.5 p
Total possible for part 2:	**3.0 p**

1.3.3 Part 3 (1 p)

Collection of data and preliminary computations for variables

Finding z and Q: Both Methods	
student has collected a dataset that could be used to find b and d	0.1 p
data set shows E	0.1 p
data set shows initial and final x and y	0.2 p
data set correctly computes b	0.1 p
data set correctly computes d or δ	0.1 p
data set has at least 8 measurements (0.4p); data set has at least 4 measurements (0.3p)	0.4 p
Total possible for part 3:	**1.0 p**

Notes: E can be measured in Joules or eV; if a student only records voltage and uses it throughout the problem in place if E, there is no penalty. If a student fails to properly record both the initial and final values for x and y in each measurement, then they do not get the 0.2 p above. If they do a set of measurements where x or y initial is held constant, they only need to record it once, but they must make it clear where it applies. Results of "miss" should be recorded, but do not count toward the measurement count of 8 or 4. There is no penalty for failing to record "miss".

1.3.4 Part 4 (2.5 p)

Selection of an approach to solve, deriving the math and physics; and developing a plot. This section is not concerned with the accuracy of the z_Q and Q results; that will be assessed in part 5.

Finding z and Q: Method 1 or 1a	
derive correct relationship between d (or δ) and b (1.0p); if there is exactly one math/geometry error (0.6p); if there are exactly two such errors (0.2p); if there are no such errors but exactly one physics error (0.4p); neglecting the correction in method 1a is a (-0.5p) deduction	1.0 p
recognise that bE must be constant	0.3 p
a plot exists of b versus d (or δ)	0.6 p
Use slope of plot to find $\cos\theta$. Don't worry about accuracy here; only that they try.	0.3 p
Use intercept of plot to find z. Don't worry about accuracy here; only that they try.	0.3 p
Total for part 4:	**2.5 p**

Finding z and Q: Method 2 or 2a	
derive correct relationship between E and d or δ (1.3p); if there is exactly one math/geometry error (1.0p); if there are exactly two such errors (0.4p); if there are no such errors but exactly one physics error (0.7p)	1.3 p
a plot exists of $2E/d$ (or $2E/\delta$) versus E^2	0.6 p
Use slope and intercept of plot to find γ (or equivalent). Don't worry about accuracy here; only that they try.	0.3 p
Use slope and intercept of plot to find z (or equivalent). Don't worry about accuracy here; only that they try.	0.3 p
Total for part 4:	**2.5 p**

Finding only the product zQ: Method 3	
derive correct relationship between E and δ (1.3p); if there is exactly one math/geometry error (1.0p); if there are exactly two such errors (0.4p); if there are no such errors but exactly one physics error (0.7p)	1.3 p
a plot exists of δ versus $1/E$, or δ versus $1/b$, or E versus $1/b$; the remaining variable being held constant.	0.6 p
Use slope of plot to find Qz (or equivalent). Don't worry about accuracy here; only that they try.	0.3 p
Total for part 4:	**2.2 p**

For the plots, worth up to 0.6p, deduct -0.1p for each axis without a label, -0.1p for each axis without a scale or scale done incorrectly, -0.1p for each incorrectly plotted point, -0.1p for best fit line not being straight, but the total plot score cannot go negative.

For students who solve the linear equation algebraically and don't show a plot: there needs to be a clear indication that they used linear regression (0.2p); a computed correlation coefficient or equivalent to assess the goodness/accuracy of fit (0.2p); a clear assessment that a linear fit (as opposed to a quadratic, or exponential, or other) was indeed merited (0.2p).

A student attempting only method 3 cannot get full marks for this part.

Student who attempt more than one method will ordinarily only receive the marks for the method that yields them the higher score.

1.3.5 Part 5 (2.5 p)

Assessing the accuracy of the result for z_Q and Q

Finding z and Q: Methods 1 or 2	
$\lvert z\rvert$ in range $11 \to 12$ cm (1.0p); in range $10 \to 13$ cm (0.7p); in range $8 \to 14$ cm (0.4p); in range $6 \to 20$ cm (0.1p)	1.0 p
Q is negative!	0.1 p
$\lvert Q\rvert$ in range $70 \to 100$ pC (0.9p); in range $50 \to 150$ pC (0.7p); in range $10 \to 500$ pC (0.4p); in range $1 \to 1000$ pC (0.1p)	0.9 p
z and Q values each within two student stated error of $\lvert z\rvert = 11.5$ cm and $\lvert Q\rvert = 86$ pC (0.3p); within two stated errors for one value (0.2p); error stated, but out of bounds for both (0.1p)	0.3 p
Statement of error for both z and Q clearly reflected and consistent in graphical picture or math, addresses or comments on both random error and systematic error of approxmation (0.2p); statement of error only concerned with random or systematic, but not both (0.1p)	0.2 p
Total possible for part 5:	**2.5 p**

Finding only zQ: any method	
Q is negative!	0.1 p
$\lvert zQ\rvert$ in range $9.7 \to 10.1$ pCm (0.9p); in range $9.5 \to 10.3$ pCm (0.7p); in range $9 \to 11$ pCm (0.4p); in range $5 \to 20$ pCm (0.1p)	0.9 p
zQ values within one student stated error of $\lvert zQ\rvert = 9.9$ pCm (0.3p); within two stated errors (0.2p); error stated, but out of bounds more than twice (0.1p)	0.3 p
Statement of error for zQ clearly reflected and consistent in graphical picture or math, addresses or comments on both random error and systematic error of approximation (0.2p); statement of error only concerned with random or systematic, but not both (0.1p)	0.2 p
Total possible for part 5:	**1.5 p**

The sources for error are (1) beam spread of 0.5mm, (2) pixel resolution of 1mm, (3) approximations for defining the tangent, (4) approximations for final trajectory approaching the asymptote, (5) approximations for intersections of the asymptote. The first two are random error; the last three are systematic.

A student that computes z, Q, and zQ should be assessed for each of the three (1.0p each) for accuracy against expected value, but will only receive the highest two results.

2 Black box

Let the tension forces in the two springs be F_1 and F_2, respectively. Let the height of the ceiling of the box be y_1 and let the heights of the masses be y_2 and y_3 (we will assume for simplicity that the masses have zero height). Let a_1, a_2, a_3 be the respective accelerations. We get the following equations of motion (ignoring all drag forces):

$$m_1a_1 = F - F_1 - m_1g$$

$$m_2a_2 = F_1 - F_2 - m_2g$$

$$m_3a_3 = F_2 - m_3g$$

Since the springs are nonlinear, $F_1 \neq k_1(y_1 - y_2)$ and $F_2 \neq k_2(y_2 - y_3)$ in general, but we know that for small displacements near equilibrium $k_1 = \frac{\Delta F_1}{\Delta(y_1-y_2)}$ and $k_2 = \frac{\Delta F_2}{\Delta(y_2-y_3)}$.

2.1 Finding $m_1 + m_2 + m_3$

When the system is at rest and at equilibrium, then the force needed to hold the box is the total gravitational force $F_0 = (m_1 + m_2 + m_3)g$ (we can get the same result if we plug in $a_1 = a_2 = a_3 = 0$ to the equations of motion).

To measure F_0, we find the value of F when the box is at rest ($a_1 = 0$). We notice that the force is constant which means that the system is initially already in equilibrium.

After averaging 10 first values we get $F_0 \approx 14.774\,\mathrm{N}$ and

$$m_1 + m_2 + m_3 = \frac{F_0}{g} = \frac{14.774\,\mathrm{N}}{9.81\,\mathrm{N/kg}} \approx 1.506\,\mathrm{kg}.$$

Exact answer: 1.506 kg.

Finding $m_1 + m_2 + m_3$		
1a	Notice that $F = g\sum m_i$ when $a_1 = 0$	0.5
1b	Measurement for F_0 ($14.77 \pm 0.10\,\mathrm{N}$) 0 points for only having a measurement without an idea how to use it	0.3
1c	$\sum m_i$ in range $1.51 \pm 0.01\,\mathrm{kg}$	0.2
	Total:	**1.0**

Note: Measurement for F_0 is needed for full points, even if $\sum m_i$ is correct. Solutions with raw data missing get 0.7 points. Solutions using $\frac{F_0}{g}$ implicitly as the sum of masses get 0.2 points from 1c.

2.2 Finding m_1

We get from the first equation of motion that $F = m_1a_1 + m_1g + F_1$. The spring force F_1 depends only on the positions (and is the same at the beginning of every experiment), so the force F at the beginning of the experiment depends only on the acceleration.

Therefore, we can measure how much the initial force changes with acceleration to get m_1. We will use maximum acceleration ($30\,\mathrm{m/s^2}$) for highest accuracy. The average of three values is $F_{30} \approx 40.487\,\mathrm{N}$, so

$$m_1 = \frac{F_{30} - F_0}{a} = \frac{(40.487 - 14.774)\,\mathrm{N}}{30\,\mathrm{m/s^2}} \approx 0.857\,\mathrm{kg}.$$

We also conclude that $m_2 + m_3 = 0.649\,\mathrm{kg}$.

(To even increase accuracy, one could compare F_{30} and F_{-30} and find their difference.)

Exact answer: 0.857 kg.

Finding m_1		
2a	$F = m_1a_1 + m_1g + F_1$ or any equivalent equation of motion (max points even if F_1 has been incorrectly substituted with $k_1(y_1 - y_2)$)	0.5
2b	Idea that $m_1 = \frac{\Delta F}{\Delta a_1}$	0.5
2c	Using $\Delta a_1 \geq 10\,\mathrm{m/s^2}$	0.2
2d	m_1 in range $0.857 \pm 0.002\,\mathrm{kg}$ m_1 in range $0.857 \pm 0.010\,\mathrm{kg}$ m_1 in range $0.857 \pm 0.050\,\mathrm{kg}$	0.8 0.6 0.3
	Total:	**2.0**

Note: Using free-fall ($a_1 = -g$) without repeated measurements gets 0 points from 2c. Full points are given if $\Delta a_1 < 10\,\mathrm{m/s^2}$ but several measurements are used that give at least as good accuracy overall.

2.3 Finding k_1

2.3.1 Method 1: Change of force after a fast movement of the box

We will quickly accelerate and then decelerate the box (to avoid drag forces). When we change the height of the box quickly and the time is short enough, we can assume that the second mass stays approximately at rest.

(Formally, if $\Delta y_1 = \frac{a_1}{2}t^2$, then $m_2a_2 = k_1\Delta y_1 - k_1\Delta y_2 - \Delta F_2 \leq k_1\Delta y_1$, therefore $a_2 \leq \frac{k_1}{2m_2}t^2 \cdot a_1$. The assumption holds if $\frac{k_1}{2m_2}t^2 \ll 1$.)

Therefore, if we accelerate the box with acceleration a_1 for time t and then with $-a_1$ for time t, then $\Delta F \approx k_1\Delta y_1 = k_1a_1t^2$.

To have the best accuracy we will do two experiments with $a_1 = 30\,\mathrm{m/s^2}$ and $a_1 = -30\,\mathrm{m/s^2}$, respectively. We will use $t = 0.01\,\mathrm{s}$ (smallest time possible). We will also repeat each experiment 5 times. After averaging the results, we get the forces at $2t = 0.02\,\mathrm{s}$ to be $F_\mathrm{u} \approx 14.890\,\mathrm{N}$ and $F_\mathrm{d} \approx 14.652\,\mathrm{N}$. Therefore

$$k_1 \approx \frac{F_\mathrm{u} - F_\mathrm{d}}{2a_1t^2} = \frac{(14.890 - 14.652)\,\mathrm{N}}{2 \cdot 30\,\mathrm{m/s^2} \cdot (0.01\,\mathrm{s})^2} \approx 39.7\,\mathrm{N/m}$$

Exact answer: 39.2 N/m.

Finding k_1, Method 1		
3.1a	Idea to use method	0.5
3.1b	Notice that if t is small, then $\Delta y_1 \gg \Delta y_2$	0.5
3.1c	Correct formula for k_1	0.5
3.1d	At least 3 measurements	0.1
3.1e	$\Delta a_1 \geq 30\,\mathrm{m/s^2}$	0.2
3.1f	$2t \leq 0.08\,\mathrm{s}$	0.2
3.1g	k_1 in range $39.2 \pm 1.0\,\mathrm{N/m}$ k_1 in range $39 \pm 4\,\mathrm{N/m}$ k_1 in range $39 \pm 8\,\mathrm{N/m}$ k_1 in range $39 \pm 15\,\mathrm{N/m}$	1.0 0.7 0.4 0.2
	Total:	**3.0**

2.3.2 Method 2: Change of force while accelerating the box

We will accelerate the box with constant acceleration a_1 and similarly as in the previous method conclude that $\Delta F \approx \frac{k_1 a_1}{2} t^2$ when t is small. This method is, however, less accurate than the previous method because drag force is nonnegligible for large values of a_1 but the resolution of ΔF is small for small values of a_1.

Choosing, for example, $a_1 = 30\,\mathrm{m/s^2}$, $t = 0.02\,\mathrm{s}$ and averaging 5 values gives $F_{t=0} \approx 40.482\,\mathrm{N}$ and $F_{t=0.02} \approx 40.792\,\mathrm{N}$ and

$$k_1 \approx \frac{2\Delta F}{a_1 t^2} = \frac{2 \cdot (40.792 - 40.482)\,\mathrm{N}}{30\,\mathrm{m/s^2} \cdot (0.02\,\mathrm{s})^2} \approx 51.7\,\mathrm{N/m}$$

The answer is $\sim 30\%$ larger than the correct answer because the drag force of the box at $t = 0.02\,\mathrm{s}$ is approximately 0.08 N.

A better choice would be $a_1 = 5\,\mathrm{m/s^2}$ and $t = 0.02\,\mathrm{s}$. Averaging 5 values gives $F_{t=0} \approx 19.058\,\mathrm{N}$ and $F_{t=0.02} \approx 19.100\,\mathrm{N}$ and

$$k_1 \approx \frac{2\Delta F}{a_1 t^2} = \frac{2 \cdot (19.100 - 19.058)\,\mathrm{N}}{5\,\mathrm{m/s^2} \cdot (0.02\,\mathrm{s})^2} \approx 42.0\,\mathrm{N/m}$$

The drag force has a much smaller effect (approximately 0.002 N).

Finding k_1, Method 2		
3.2a	Idea to use method	0.5
3.2b	Notice that if t is small, then $\Delta y_1 \gg \Delta y_2$	0.5
3.2c	Correct formula for k_1	0.5
3.2d	At least 3 measurements	0.1
3.2e	$2\,\mathrm{m/s^2} \leq a_1 \leq 10\,\mathrm{m/s^2}$	0.2
3.2f	$t \leq 0.08\,\mathrm{s}$	0.2
3.1g	k_1 in range $39.2 \pm 1.0\,\mathrm{N/m}$ k_1 in range $39 \pm 4\,\mathrm{N/m}$ k_1 in range $39 \pm 8\,\mathrm{N/m}$ k_1 in range $39 \pm 15\,\mathrm{N/m}$	1.0 0.7 0.4 0.2
	Total:	**3.0**

Note: A correct answer without any justification or obtained with a physically nonsensible method gives 0 points.

2.3.3 Method 3: Estimating $y_1 - y_2$ at equilibrium and using $F_1 \approx k_1(y_1 - y_2)$

Although the springs are nonlinear, we can estimate k_1 by $k_1 \approx \frac{F_1}{y_1 - y_2}$ which would be true if the springs were perfectly linear.

At equilibrium

$$F_1 = F_0 - m_1 g \approx 14.774\,\mathrm{N} - 0.857 \cdot 9.81\,\mathrm{N} \approx 6.367\,\mathrm{N}$$

If we accelerate the box quickly downwards, then by measuring the time t for the box to collide with mass 2, we can estimate the initial value of $y_1 - y_2$ by $\Delta y_1 = \frac{a_1}{2} t^2$.

Using binary search we can find that $t \leq 0.13\,\mathrm{s}$ if $|a_1| \geq 26.7\,\mathrm{m/s^2}$ and $t \geq 0.13\,\mathrm{s}$ if $|a_2| \leq 26.6\,\mathrm{m/s^2}$.

Therefore

$$y_1 - y_2 \approx \frac{26.7\,\mathrm{m/s^2}}{2} \cdot (0.13\,\mathrm{s})^2 \approx 0.226\,\mathrm{m}$$

and

$$k_1 \approx \frac{F_1}{y_1 - y_2} = \frac{6.367\,\mathrm{N}}{0.226\,\mathrm{m}} \approx 28.2\,\mathrm{N/m}$$

This method underestimates the value both due to nonlinearity of springs and because it overestimates $y_1 - y_2$ (the actual value is 0.179 m).

Finding k_1, Method 3		
3.3a	Idea to use method	0.5
3.3b	Correctly estimate $y_1 - y_2$	0.5
3.3c	Correct formula for k_1	0.5
	Total:	**1.5**

Note: This method is worth 1.5 points since it is very inaccurate. Estimating the distance $y_1 - y_2$ without an idea how to use it gives 0 points.

Method for eye-balling k_1 from slow normal mode frequency assuming a rigid connection between m_2 and m_3 was rewarded with 0.5+0.5 points for idea and formula if significant progress was made (a reasonable value for the normal mode period and k_1 or $\frac{k_1}{m_2 + m_3}$ was found). Simply stating $T = 2\pi\sqrt{\frac{m_2 + m_3}{k_1}}$ gave 0 points.

2.4 Finding m_2, m_3 and k_2

2.4.1 Method 1: Finding natural frequencies

This method is very accurate, but needs a lot algebraic manipulation to solve for two parameters. This method could also be used to find one parameter if the other has been already found using alternative methods.

At first we will find the natural frequencies of the system when the box is at rest. Let $x_2 = \Delta y_2$ and $x_3 = \Delta y_3$ be small displacements near equilibrium. Then

$$m_2 \ddot{x}_2 = -k_1 x_2 - k_2(x_2 - x_3)$$

$$m_3 \ddot{x}_3 = k_2(x_2 - x_3)$$

The equations can be solved by taking $x_2 = A\cos(\omega t)$ and $x_3 = B\cos(\omega t)$, where A and B are constants.

(Alternatively, one can use complex numbers: $\tilde{x}_2 = Ae^{i\omega t}$ and $\tilde{x}_3 = Be^{i\omega t}$.)

We see that $\ddot{x}_2 = -\omega^2 A\cos(\omega t)$ and $\ddot{x}_3 = -\omega^2 B\cos(\omega t)$, hence

$$-m_2\omega^2 A\cos(\omega t) = -k_1 A\cos(\omega t) - k_2(A-B)\cos(\omega t)$$

$$-m_3\omega^2 B\cos(\omega t) = k_2(A-B)\cos(\omega t)$$

We see that the time dependence cancels out

$$-m_2\omega^2 A = -k_1 A - k_2 A + k_2 B$$

$$-m_3\omega^2 B = k_2 A - k_2 B$$

We get from the second equation that $B = \frac{k_2 A}{k_2 - m_3\omega^2}$, so after substituting to the first equation we get

$$-m_2\omega^2 A = -k_1 A - k_2 A + \frac{k_2^2}{k_2 - m_3\omega^2}A$$

As expected, A cancels out (because natural frequency does not depend on the amplitude of the oscillations) and we get

$$-m_2\omega^2(k_2 - m_3\omega^2) + (k_1+k_2)(k_2 - m_3\omega^2) - k_2^2 = 0$$

$$m_2 m_3\omega^4 - k_2 m_2\omega^2 - (k_1+k_2)m_3\omega^2 + k_1 k_2 = 0$$

$$\omega^4 - \left(\frac{k_2}{m_3} + \frac{k_1+k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_2 m_3} = 0$$

The solutions to this biquadratic equation are the natural angular frequencies. If we know the solutions ω_1 and ω_2, we know from the Vieta's formulas that

$$\frac{k_2}{m_3} + \frac{k_1+k_2}{m_2} = c_1$$

$$\frac{k_1 k_2}{m_2 m_3} = c_2,$$

where $c_1 = \omega_1^2 + \omega_2^2$ and $c_2 = \omega_1^2\omega_2^2$.

We find that

$$\frac{m_2}{k_1} + \frac{m_3}{k_2} + \frac{m_3}{k_1} = \frac{c_1}{c_2}$$

$$\frac{m_3}{k_2} = \frac{c_1}{c_2} - \frac{m_2+m_3}{k_1}$$

$$\frac{k_1}{m_2 c_2} = \frac{c_1}{c_2} - \frac{m_2+m_3}{k_1}$$

$$m_2 = \frac{k_1^2}{c_1 k_1 - c_2(m_2+m_3)}$$

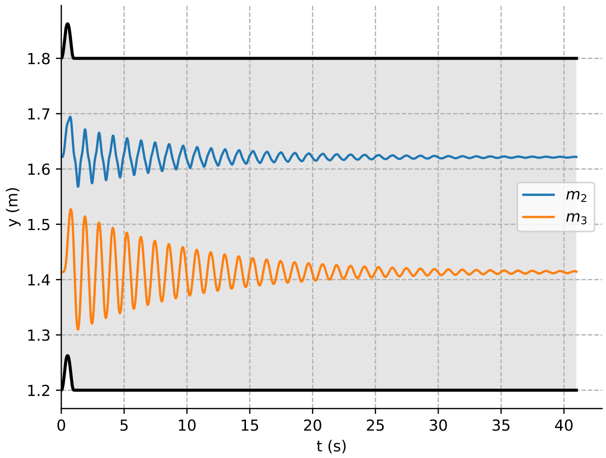
This equation allows us to find m_2. After this, it is easy to also find m_3 and k_2.

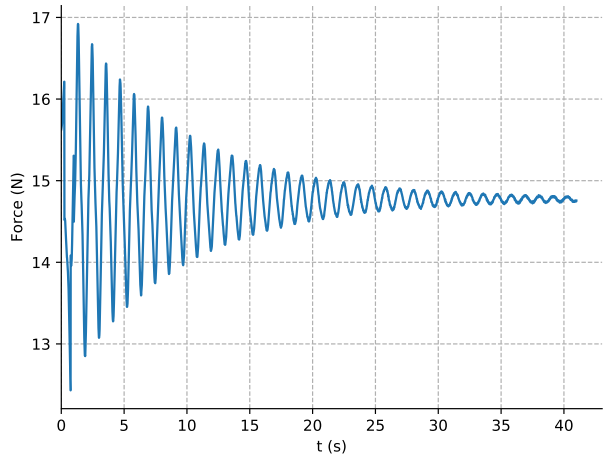
To find the natural frequencies, one can oscillate the box with different frequencies, stop oscillating and look at how force changes in time. Using trial and error we can get two estimates $T_1 \approx 1\,\mathrm{s}$ and $T_2 \approx 0.4\,\mathrm{s}$.

To find the smaller frequency, we can, for example, give the box a pulse with 1 s duration.

We want to be sure that the amplitude of the oscillations is small enough when we measure the period (to avoid nonlinearity of springs). We find

$$T_1 = \frac{(34.70 - 20.27)\,\mathrm{s}}{13} \approx 1.11\,\mathrm{s}.$$





Similarly, we can amplify the larger natural frequency by oscillating the box or giving a shorter pulse.

We find

$$T_2 = \frac{(20.00 - 9.94)\,\mathrm{s}}{27} \approx 0.373\,\mathrm{s}.$$

Therefore

$$\omega_1^2 = \left(\frac{2\pi}{T_1}\right)^2 \approx 32.04\,\mathrm{Hz}^2$$

$$\omega_2^2 = \left(\frac{2\pi}{T_1}\right)^2 \approx 283.8\,\mathrm{Hz}^2$$

We can then find m_2 by calculating c_1 and c_2:

$$c_1 = \omega_1^2 + \omega_2^2 \approx 315.8\,\mathrm{Hz}^2$$

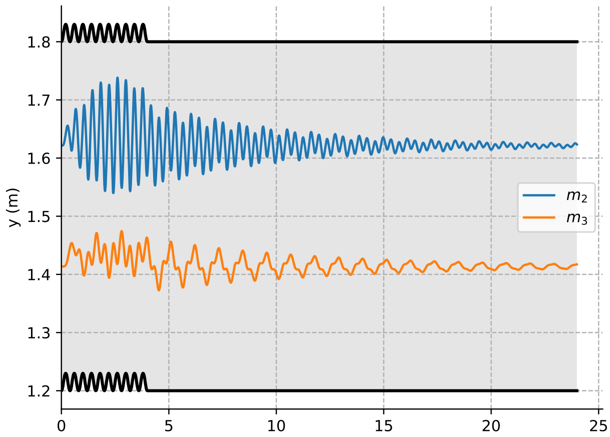
$$c_2 = \omega_1^2\omega_2^2 \approx 9093\,\mathrm{Hz}^4$$

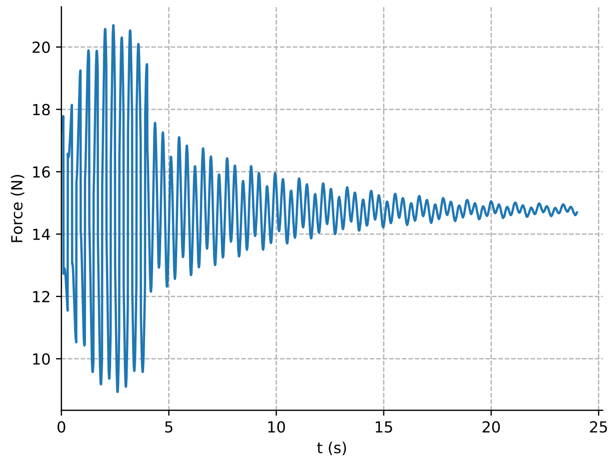
$$m_2 = \frac{k_1^2}{c_1 k_1 - c_2(m_2+m_3)} \approx 0.238\,\mathrm{kg}$$

$$m_3 = 0.649\,\mathrm{kg} - 0.238\,\mathrm{kg} = 0.411\,\mathrm{kg}$$

$$k_2 = \frac{c_2 m_2 m_3}{k_1} \approx 22.4\,\mathrm{N/m}$$

Exact answers: $m_2 = 0.236\,\mathrm{kg}$, $m_3 = 0.413\,\mathrm{kg}$, $k_2 = 22.6\,\mathrm{N/m}$.





2.4.2 Method 2: Fast pulse

Similarly as in method 1 for finding k_1, we quickly accelerate the box with acceleration a_1 for time t and then decelerate with acceleration $-a_1$ for time t. If t is small, then y_2 does not change much while moving the box, so $\Delta F_1 = \Delta F \approx k_1 \Delta y_1$.

We also know that after the pulse when mass 2 starts to move, for a short time y_3 does not change much.

Therefore, $m_2 a_2 = \Delta F_1 - \Delta F_2 \approx \Delta F_1$ before mass 2 starts to significantly move, and $m_2 a_2 \approx k_1 \Delta y_1 - k_1 \Delta y_2 - k_2 \Delta y_2 = k_1 \Delta y_1 - (k_1 + k_2)\Delta y_2$ before mass 3 starts to significantly move.

Therefore, if t is small, then right after time t:

$$F - F_0 = \Delta F_1 \approx m_2 a_2 = m_2 \frac{d^2 y_2}{dt^2} = -\frac{m_2}{k_1}\frac{d^2 F_1}{dt^2} = -\frac{m_2}{k_1}\frac{d^2 F}{dt^2}.$$

This method is less accurate than the previous method, it does many approximations, ignores drag forces and resolution of F is small. It might be possible to make this method more accurate by taking the initial estimate for m_2 and then estimating Δy_2 while the box is accelerated to get a better estimate.

Theoretically it is also possible to find k_2, although it is even less accurate. When $\frac{d^2F}{dt^2} = 0$, then $a_2 = 0$, which means that $k_1 \Delta y_1 \approx (k_1 + k_2)\Delta y_2$.

We will use $a_1 = 30\,\mathrm{m/s^2}$ for highest accuracy. A good trade-off between resolution of F and small t seems to be $t = 0.05\,\mathrm{s}$. Since we need to find the second derivative and need a lot of accuracy, we will average the values of 10 measurements. The results are shown in the table.

Time (s)	F (N)	$\frac{dF}{dt}$ (N/s)
0.10	12.572	
0.11	12.781	20.9
0.12	13.013	23.2
0.13	13.268	25.5
0.14	13.533	26.5
0.15	13.811	27.8
0.16	14.078	26.7
0.17	14.351	27.3
0.18	14.599	24.8

We estimate that $\frac{d^2F}{dt^2} \approx 230\,\mathrm{N/s^2}$ at $t = 0.11\,\mathrm{s}$. Therefore

$$m_2 \approx -\frac{(F - F_0)k_1}{\frac{d^2F}{dt^2}} = -\frac{(12.781 - 14.774)\,\mathrm{N} \cdot 39.7\,\mathrm{N/m}}{230\,\mathrm{N/s^2}}.$$

$$m_2 \approx 0.344\,\mathrm{kg}.$$

We also estimate that $\frac{d^2F}{dt^2} = 0$ at $t \approx 0.15\,\mathrm{s}$ when $F = 13.811\,\mathrm{N}$. We know that $\Delta y_1 = a_1 t^2 = -30\,\mathrm{m/s^2} \cdot (0.05\,\mathrm{s})^2 = -0.075\,\mathrm{m}$.

At $F = 13.811\,\mathrm{N}$,

$$\Delta(y_1 - y_2) \approx \frac{(13.811 - 14.774)\,\mathrm{N}}{39.7\,\mathrm{N/m}} \approx -0.024\,\mathrm{m},$$

where we get $\Delta y_2 \approx 0.051\,\mathrm{m}$.

Since $k_1 \Delta y_1 \approx (k_1 + k_2)\Delta y_2$,

$$k_2 \approx k_1\left(\frac{\Delta y_1}{\Delta y_2} - 1\right) \approx 18.4\,\mathrm{N/m}$$

	Finding m_2, m_3, k_2, Any method	
4a	Correct method	0.5
4b	Correct equations allowing to solve for the values m_2, m_3, k_2	0.5
	Correct equations allowing to solve only for m_2 and m_3	0.3
4c	Necessary measurements	1.0
	If only natural frequencies (periods) are found without a plan on how to use them:	
	T_1 in range $1.11 \pm 0.02\,\mathrm{s}$	0.3
	T_1 in range $1.11 \pm 0.10\,\mathrm{s}$	0.1
	T_2 in range $0.373 \pm 0.005\,\mathrm{s}$	0.3
	T_2 in range $0.373 \pm 0.050\,\mathrm{s}$	0.1
4d	k_2 in range $22.6 \pm 0.5\,\mathrm{N/m}$	1.0
	k_2 in range $22.6 \pm 1.0\,\mathrm{N/m}$	0.8
	k_2 in range $23 \pm 3\,\mathrm{N/m}$	0.6
	k_2 in range $23 \pm 6\,\mathrm{N/m}$	0.4
4e	m_2 in range $0.236 \pm 0.010\,\mathrm{kg}$ or m_3 in range $0.413 \pm 0.010\,\mathrm{kg}$	0.9
	m_2 in range $0.236 \pm 0.020\,\mathrm{kg}$ or m_3 in range $0.413 \pm 0.020\,\mathrm{kg}$	0.6
	m_2 in range $0.236 \pm 0.050\,\mathrm{kg}$ or m_3 in range $0.413 \pm 0.050\,\mathrm{kg}$	0.3
4f	Correctly calculate m_3 using m_2 or vice versa given any points received in 4e	0.1
	Total:	**4.0**

Note: Equations of motion for mass 2 and 3 give 0 points. Getting a correct biquadratic equation for ω^2

gives 0.5 points from 4a, points are given for 4b only if k_2, m_2, or m_3 is correctly expressed from the biquadratic equation taking k_1, m_2+m_3, ω_1 and ω_2 as the only known parameters. Partial points can be given for getting an equation for ω^2 (0.4 p for slightly wrong result, 0.2 p for setting up the determinant). Finding T_1 and T_2 give 0.3/0.1 points even with a plan to use them to solve a system of equations but without an idea how. Having a biquadratic equation for ω^2 counts as "a plan" and in this case, finding T_1 and T_2 will each give 0.5/0.2 points with the same error tolerances. A correct answer without any justification or obtained with a physically nonsensible method gives 0 points.

2.4.3 Method 3: Estimating $\frac{k_2}{m_3}$ by estimating $y_2 - y_3$ at equilibrium and using $F_2 \approx k_2(y_2 - y_3)$

After finding $y_1 - y_2$ using method 3 to find k_1, we can similarly estimate $y_3 - (y_1 - a)$, where $a = 0.6\,\mathrm{m}$, by quickly accelerating the box upwards. This method assumes that the masses have negligible height (which is true).

Again, using binary search, we find that the time for collision is $t = 0.13\,\mathrm{s}$ at $a_1 = 25.6\,\mathrm{m/s^2}$. Thus

$$y_3 - (y_1 - a) \approx \frac{a_1}{2}t^2 \approx 0.216\,\mathrm{m}$$

$$y_2 - y_3 = a - (y_1 - y_2) - (y_3 - y_1 + a)$$

$$y_2 - y_3 \approx 0.6\,\mathrm{m} - 0.226\,\mathrm{m} - 0.216\,\mathrm{m} \approx 0.158\,\mathrm{m}$$

$$\frac{k_2}{m_3} \approx \frac{g}{y_2 - y_3} \approx 62.1\,\mathrm{N/(kg\,m)}$$

The actual values are $y_2 - y_3 = 0.208\,\mathrm{m}$ and $\frac{k_2}{m_3} = 54.7\,\mathrm{N/(kg\,m)}$.

Estimating k_2/m_3		
4.1a	Idea to use method	0.5
4.1b	Correctly estimate $y_3 - y_1 + a$	0.5
4.1c	Correct formula for k_2/m_3	0.2
4.1d	k_2/m_3 in range $55 \pm 10\,\mathrm{N/(kg\,m)}$	0.3
	Total:	**1.5**

Note: Estimating the distance $y_3 - y_1 + a$ without an idea how to use it gives 0 points.