Problem 2 : Solution/marking scheme – Nonlinear Dynamics in Electric Circuits *(10 points)*

Part A. Stationary states and instabilities (3 points)

Solution A1: **[0.4]**

By looking at the $I - V$ graph, we obtain

$$R_{\text{off}} = 10.0\,\Omega, \quad \textbf{0.1}$$

$$R_{\text{on}} = 1.00\,\Omega, \quad \textbf{0.1}$$

$$R_{\text{int}} = 2.00\,\Omega, \quad \textbf{0.1}$$

$$I_0 = 6.00\,A. \quad \textbf{0.1}$$

Solution A2: **[1]**

Kirchoff law for the circuit (U is the voltage of the bi-stable element):

$$\mathcal{E} = IR + U \quad \textbf{0.2}$$

This yields

$$I = \frac{\mathcal{E} - U}{R} \quad \textbf{0.1}$$

Hence, stationary states of the circuit are intersections of the line defined by this equation and the $I - V$ graph of X. **0.2**

For $R = 3.00\,\Omega$, one always gets exactly one intersection. **0.2**
For $R = 1.00\,\Omega$, one gets 1, 2 or 3 intersections depending on the value of $\mathcal{E}$. **0.3**

Solution A3: **[0.6]**

The stationary state is on the intermediate branch, one can thus use the corresponding equation: **0.2**

$$\begin{aligned} I_{\text{stationary}} &= \frac{\mathcal{E} - R_{\text{int}} I_0}{R - R_{\text{int}}} && \textbf{0.1} \\ &= 3.00\,A && \textbf{0.1} \end{aligned}$$

$$\begin{aligned} U_{\text{stationary}} &= R_{\text{int}}(I_0 - I) && \textbf{0.1} \\ &= 6.00\,V && \textbf{0.1} \end{aligned}$$

Solution A4: **[1]**

The Kirchoff law for the circuit reads

$$\mathcal{E} = IR + U_X + L\frac{dI}{dt} = IR + (I_0 - I)R_{\text{int}} + L\frac{dI}{dt}$$ **0.3**

This implies

$$L\frac{dI}{dt} = \mathcal{E} - I_0 R_{\text{int}} - (R - R_{\text{int}})I$$ **0.2**

The right relative sign of dI/dt is of importance.
If $I > I_{\text{stationary}}$, we have $dI/dt < 0$ and I decreases. **0.2**
If $I < I_{\text{stationary}}$, we have $dI/dt > 0$ and I increases. **0.2**
We conclude that the stationary state is stable. **0.1**

Note: The checkbox gives 0.1 points if "stable" is checked, regardless of the previous reasoning (also if there is nothing). A wrong reasoning leading to check the "unstable" option doesn't however give any point for the checkbox.

Part B. Thyristors in physics and engineering: radio transmitter (5 points)

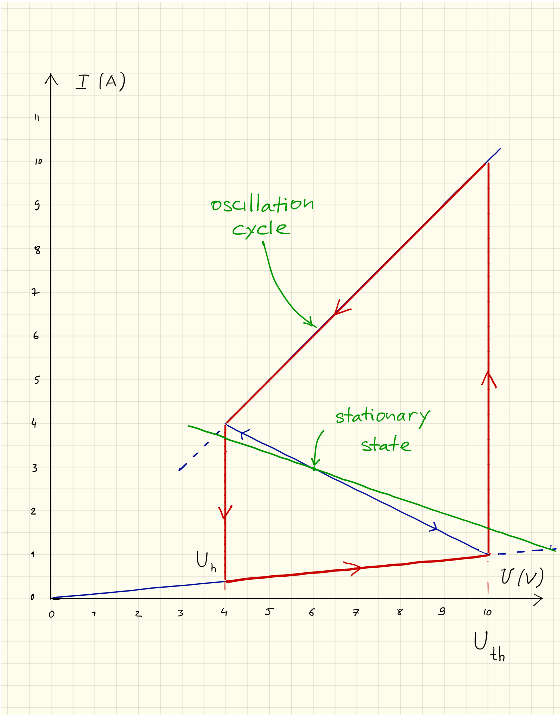
Solution B1: **[1.8]**

Since there is a stationary state on the intermediate branch, the cycle cannot go through this branch (otherwise, it would have stopped there). Hence, the cycle should only include the switched on and switched off branches. **0.6**
Switched on and switched off branches have an $I - V$ characteristics of a resistor with $R_{\text{on}} = 1.00\,\Omega$ and $R_{\text{off}} = 10.0\,\Omega$ (see **A1**). If these branches continued indefinitely, there would have been stationary states with

$$U_{\text{on}} = \frac{R_{\text{on}}}{R + R_{\text{on}}}\mathcal{E} = 3.75\,\text{V}, \quad U_{\text{off}} = \frac{R_{\text{off}}}{R + R_{\text{off}}}\mathcal{E} = 11.5\,\text{V}.$$

Since these stationary states would be unique stationary states, they must be stable. **0.4**
On the switched off branch, the system moves to the right. **0.2**
On the switched on branch, the system moves to the left. **0.2**
The cycle has the following shape: the system moves to the left along the switched on branch, at U_h it jumps to the switched off branch,then moves to the right and at U_{th} jumps to the switched on branch again. **0.4**

Note: Should a student give a correct graph without any justification, then only the points for the graph are given (up to *0.8 points*)



Solution B2: **[1.9]**

Since the non-linear element is oscillating between the switched on and switched off branches we can put $U_X = \rho I_X$, where ρ takes the value R_{on} or R_{off} depending on the branch. Hence, we get the Kirchoff law

$$R\rho C\,\frac{dI_X}{dt} = \mathcal{E} - (\rho + R)I_X$$ **0.5**

This equation admits a unique solution

$$I_X(t) = \frac{\mathcal{E}}{\rho + R} + \left(I_X(0) - \frac{\mathcal{E}}{\rho + R}\right)\, e^{-\frac{\rho+R}{\rho RC}\,t}.$$

For the voltage drop on the non-linear element we obtain

$$U_X(t) = \frac{\rho}{\rho + R}\,\mathcal{E} + \left(U_X(0) - \frac{\rho}{\rho + R}\,\mathcal{E}\right)\, e^{-\frac{\rho+R}{\rho RC}\,t}$$

There are 0.5 points distributed as follow for $U_X(t)$:

- Correct exponential **0.2**
- Correct constant term $(t \to \infty)$ **0.1**
- Correct coefficient in front of the exponential **0.1**
- Correct equation for $U_X(t)$ **0.1**

Time spent by the system on the switched on branch during one cycle:

$$t_{\text{on}} = \frac{R_{\text{on}}R}{R_{\text{on}} + R}\,C\,\log\left(\frac{U_{\text{th}} - U_{\text{on}}}{U_{\text{h}} - U_{\text{on}}}\right) = 2.41 \cdot 10^{-6}\,s,$$ **0.4**

Time spent by the system on the switched off branch during one cycle:

$$t_{\text{off}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log \left(\frac{U_{\text{off}} - U_{\text{h}}}{U_{\text{off}} - U_{\text{th}}} \right) = 3.71 \cdot 10^{-6}\, s. \quad \textbf{0.4}$$

The total period of oscillations:

$$T = t_{\text{on}} + t_{\text{off}} = 6.12 \cdot 10^{-6}\, s \quad \textbf{0.1}$$

Solution B3: **[0.7]**

Neglect the energy consumed on the switched off branch. The energy consumed on the switched on branch during the cycle is estimated by

$$E = \frac{1}{R_{\text{on}}} \left(\frac{U_h + U_{th}}{2} \right)^2 t_{\text{on}} = 1.18 \cdot 10^{-4}\,\text{J}. \quad \textbf{0.4}$$

For the power, this gives an estimate of

$$P \sim \frac{E}{T} = 19.3\,\text{W}. \quad \textbf{0.2}$$

An order of magnitude is 10 W. **0.1**

Note: For any other alternative solutions (highest value on the upper branch, integration or any other reasonable method), 0.4 points are awarded for the calculation of the energy dissipation, 0.2 for the calculation of the power and 0.1 for estimating the order of magnitude.

Solution B4: **[0.6]**

The wave length of the radio signal is given by $\lambda = cT = 1.82 \cdot 10^3$ m. **0.2**
The optimal length of the antenna is $\lambda/4$ (or $3\lambda/4, 5\lambda/4$ *etc.*) **0.3**
The only choice which is below 1 km is $s = \lambda/4 = 454$ m. **0.1**

Part C. Thyristors in biology: neuristor (2 points)

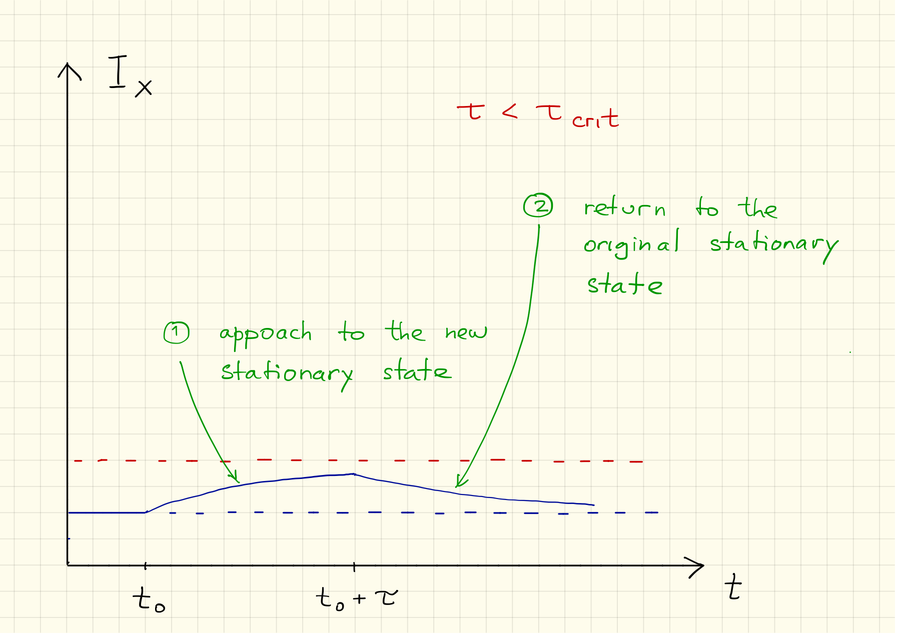
Solution C1: **[1.2]**

For $\mathcal{E}' = 12.0$ V, the steady state of the system is located on the switched off branch:

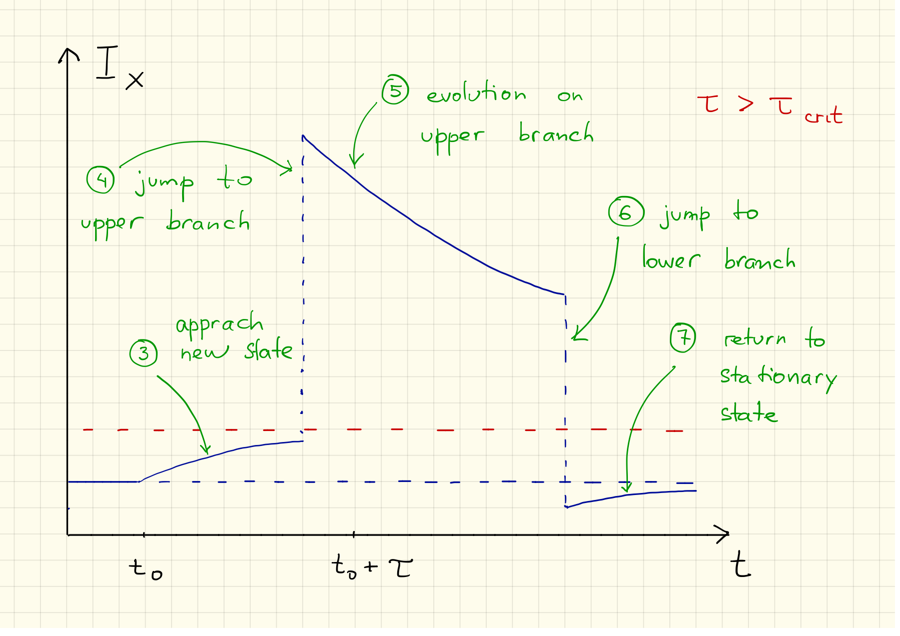
$$U' = \frac{R_{\text{off}}}{R + R_{\text{off}}} \mathcal{E}' = 9.23\,\text{V}.$$

When the voltage is increased to $\mathcal{E} = 15.0$ V, the system starts moving to the right along the switched off branch (in the same way it did in task **B**).
If the voltage drops again before the system reaches the threshold voltage, it will simply return to the stationary state.

If system reaches the threshold voltage, it will jump to the switched on branch, and it will make one oscillations (since $\tau < T$) before the voltage drops again and it returns to the stationary state.



1. Approach to the new state **0.2**
2. Return to the old stationary state **0.2**



3. Approach to the new stationary state **0.1**
4. Jump to the upper branche before τ **0.2**
5. Evolution on the upper branch **0.2**
6. Jump to the lower branche below old equilibrium **0.2**
7. Return to the old equilibrium **0.1**

Solution C2: **[0.6]**

The time needed to reach the threshold voltage is given by

$$\tau_{\text{crit}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log\left(\frac{U_{\text{off}} - U'}{U_{\text{off}} - U_{\text{th}}}\right) = 9.2 \cdot 10^{-7}\, s.$$

Note: This is the same formula as for t_{off} in task **B2**, with U_h replaced by U'.

- Correct time constant **0.2**
- Correct choice of voltages **0.2**
- Correct final formula **0.1**
- Correct numerical value **0.1**

Solution C3: **[0.2]**

Since $\tau > \tau_{\text{crit}}$, the system will make one oscillation. We conclude that the system is a neuristor. **0.2**

Note: 0.2 are given only if "Yes" is checked, regardless of the development of the other tasks (it is meant as a reward for a correct conclusion based on a correct calculation).